Balanced assignments of periodic tasks

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Introduction

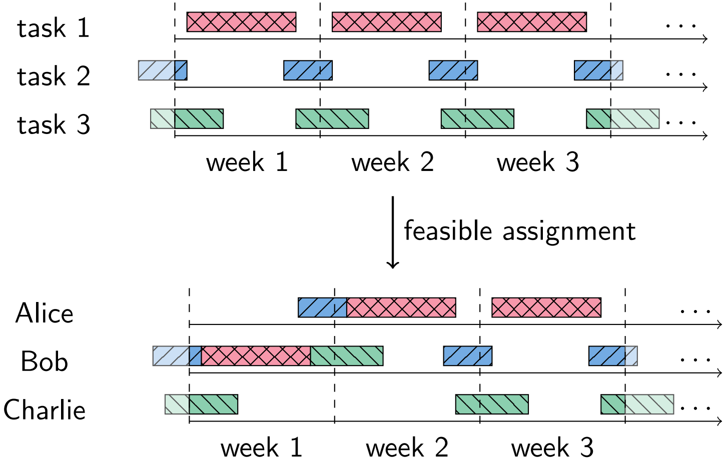
Public transports: crew scheduling and rostering with fairness between workers
→ Balanced assignment of tasks to workers

Input:

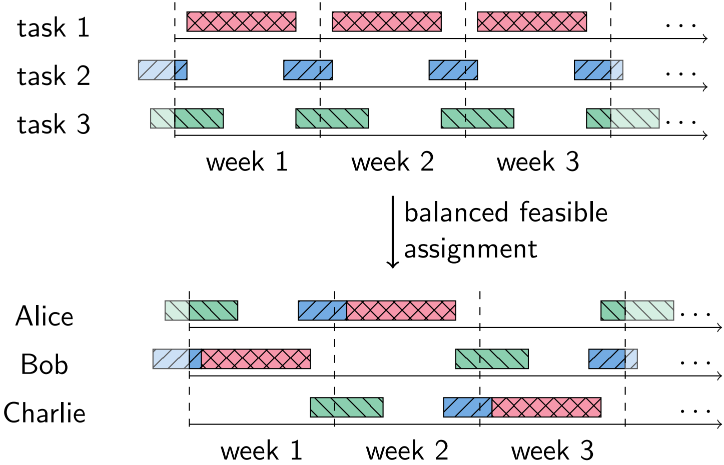
- Tasks to be performed at the same time every week
- Number of indistinguishable workers

Output: Feasible assignment of tasks to workers so that each worker performs each task with same asymptotic frequency

Feasible assignment: an example



Balanced assignment: an example



Balanced assignment: formally

Input: n tasks to be repeated every week, q workers

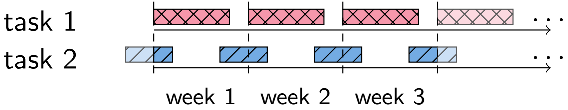
Assignment: $f\colon [n] \times \mathbb{Z}_{>0} \to [q]$, where $f(i,r) = j$ means worker j performs task i on week r

Feasible assignment: f is non overlapping

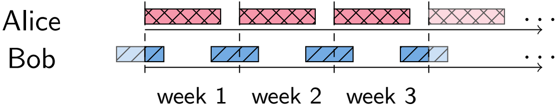
Balanced assignment:

$$\lim_{t\to+\infty} \frac{1}{t}\left|\{r \in [t]\colon f(i,r) = j\}\right| = \frac{1}{q} \qquad \forall i,j$$

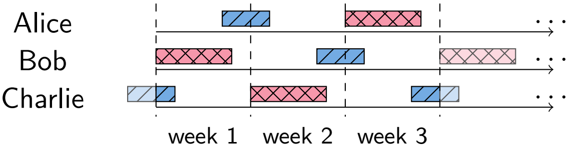
Balanced assignments: unachievable case



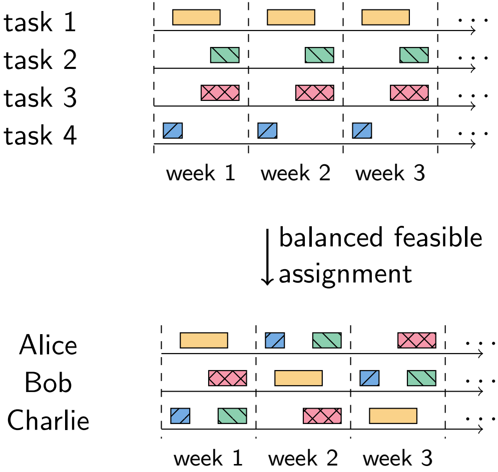
With 2 workers



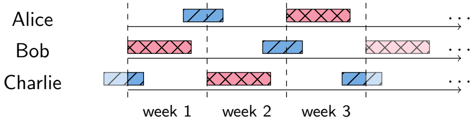
With 3 workers



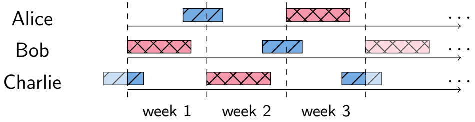
Balanced assignments: trivial case



Main results



Main results



Theorem G.–Meunier 2024+

There exists a balanced feasible assignment if and only if there exists a feasible assignment with a worker performing each task at least once.

The balanced assignment can be chosen periodic with period q.

Algorithmic proof with polynomiality results:

- Deciding existence of a balanced feasible assignment
- Computing a periodic balanced feasible assignment with period q

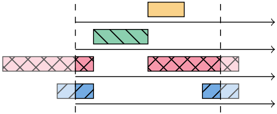
Extension

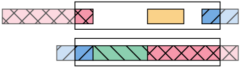
Input: n tasks to be repeated every week, q workers, $\mathcal{S}$ set of valid weekly schedules

Feasible assignment: non overlapping and induces only allowed valid schedules

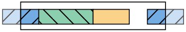
Examples:

- workers work at most H hours per week.
- workers never start two tasks labeled with a prime number in the same week.
- workers perform at most M tasks per week.
  with $M = 2$





...
valid schedules



non-valid schedule

Extension: results

Theorem G.–Meunier 2024+

If there exists a feasible assignment with a worker performing each task at least once and q tasks have to be performed at time zero, then there exists a balanced feasible assignment.

- Examples where the additionnal condition is needed

Theorem G.–Meunier 2024+

If there exists a balanced feasible assignment, then there exists a periodic balanced feasible assignment with period bounded by $q^2 q!$.

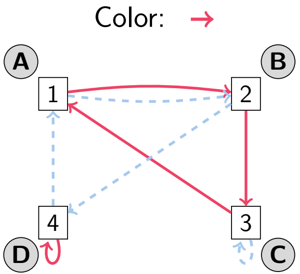
Thus the existence of a balanced feasible assignment is decidable.

Moving pebbles

Consider:

- graph $D = (V, A)$ with colored arcs, each color forming vertex disjoint cycles covering V
- one pebble on each vertex

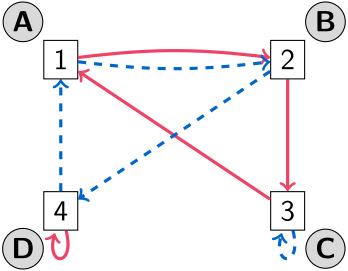
Sequence of colors defines sequence of moves of the pebbles



- Vertex = task at time zero
- Arc = valid schedule
- Pebble = worker
- Color c = partition of the tasks into schedules

Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.



Sequence of colors:

Arcs visiting:

	1→2	2→3	3→1	4→4	1⇢2	2⇢4	3⇢3	4⇢1
A	0	0	0	0	0	0	0	0
B	0	0	0	0	0	0	0	0
C	0	0	0	0	0	0	0	0
D	0	0	0	0	0	0	0	0

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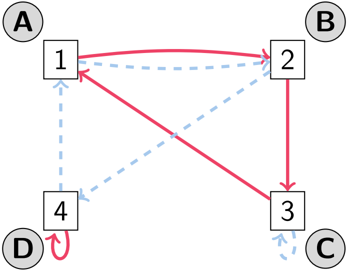

Sequence of colors:

Arcs visiting:

	1→2	2→3	3→1	4→4	1⇢2	2⇢4	3⇢3	4⇢1
A	**1**	0	0	0	0	0	0	0
B	0	**1**	0	0	0	0	0	0
C	0	0	**1**	0	0	0	0	0
D	0	0	0	**1**	0	0	0	0

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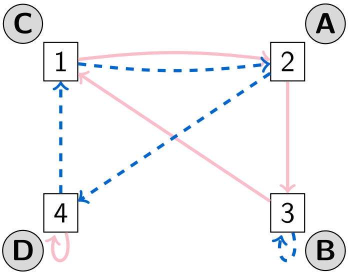

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Arcs visiting:

	1→2	2→3	3→1	4→4	1⇢2	2⇢4	3⇢3	4⇢1
A	1	0	0	0	0	**1**	0	0
B	0	1	0	0	0	0	**1**	0
C	0	0	1	0	**1**	0	0	0
D	0	0	0	1	0	0	0	**1**

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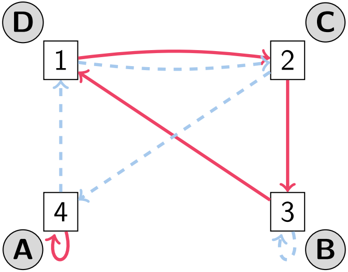

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C	0	**1**	1	0	1	0	0	0
D	**1**	0	0	1	0	0	0	1

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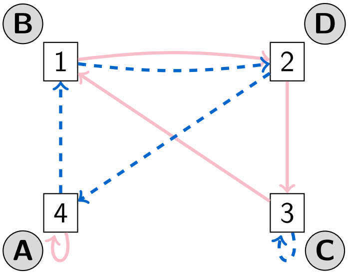

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A	1	0	0	1	0	1	0	**1**
B	0	1	1	0	**1**	0	1	0
C	0	1	1	0	1	0	**1**	0
D	1	0	0	1	0	**1**	0	1

Moving pebbles

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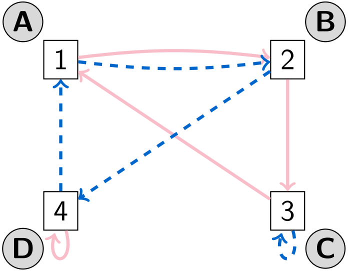

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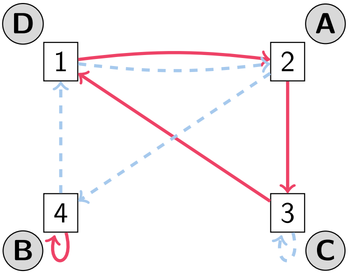
Arcs visiting:

	1→2	2→3	3→1	4→4	1⇢2	2⇢4	3⇢3	4⇢1
A	1	0	0	1	**1**	1	0	1
B	0	1	1	0	1	**1**	1	0
C	0	1	1	0	1	0	**2**	0
D	1	0	0	1	0	1	0	**2**

Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.



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A	1	**1**	0	1	1	1	0	1
B	0	1	1	**1**	1	1	1	0
C	0	1	**2**	0	1	0	2	0
D	**2**	0	0	1	0	1	0	2

Moving pebbles

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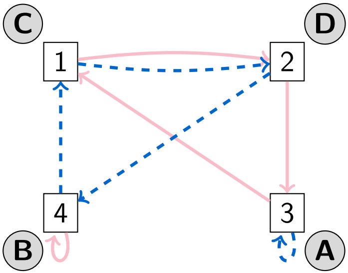

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Arcs visiting:

	1→2	2→3	3→1	4→4	1⇢2	2⇢4	3⇢3	4⇢1
A	1	1	0	1	1	1	**1**	1
B	0	1	1	1	1	1	1	**1**
C	0	1	2	0	**2**	0	2	0
D	2	0	0	1	0	**2**	0	2

Moving pebbles

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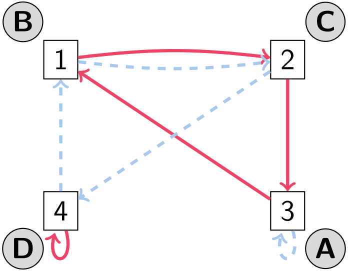

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A	1	1	**1**	1	1	1	1	1
B	**1**	1	1	1	1	1	1	1
C	0	**2**	2	0	2	0	2	0
D	2	0	0	**2**	0	2	0	2

Moving pebbles

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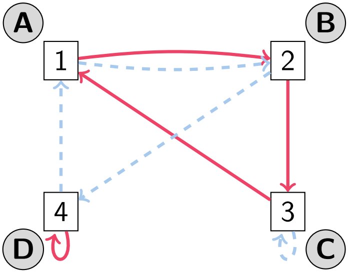

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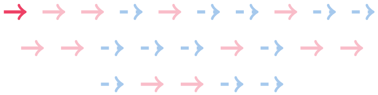

Arcs visiting:

	1 → 2	2 → 3	3 → 1	4 → 4	1 ⇢ 2	2 ⇢ 4	3 ⇢ 3	4 ⇢ 1
A	**1**	0	0	0	0	0	0	0
B	0	**1**	0	0	0	0	0	0
C	0	0	**1**	0	0	0	0	0
D	0	0	0	**1**	0	0	0	0

Back to the results

Theorem G.–Meunier 2024+

If the graph is connected, then there exists a sequence of colors making each pebble visit each arc with same asymptotic frequency.

2 proofs:

- Markov chains: non constructive
- Contructive proof with additionnal result:

Proposition G.–Meunier 2024+

The constructed sequence of colors is periodic with period bounded by $q^2 q!$.

(q: number of workers)

Open Questions

If there exists a feasible assignment with a worker performing each task at least once and q tasks have to be performed at time zero then there exists a balanced feasible assignment.

If there exists a feasible assignment with a worker performing each task at least once and no tasks have to be performed at time zero then there exists a balanced feasible assignment.

Question

Could we establish a result for the in-between case?

Question

Is the bound $q^2q!$ on the period tight for the color sequence?

Thank you