# Balanced assignments of periodic tasks

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*Joint work with Frédéric Meunier*

# Introduction

Public transports: crew scheduling and rostering with fairness between workers
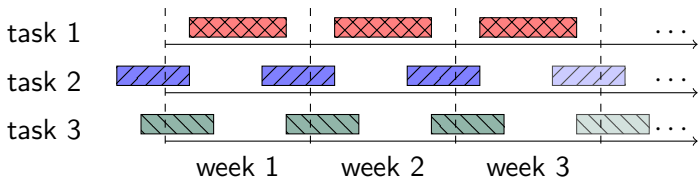$\rightarrow$ Balanced assignment of tasks to workers

Input:

- Tasks to be performed at the same time every week
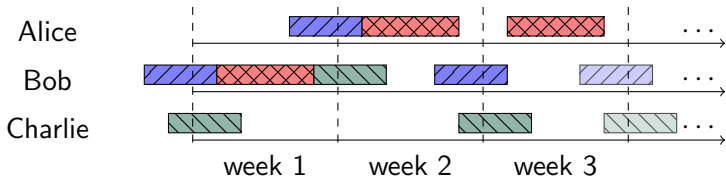- Indistinguishable workers

Output:   Assignment of tasks to workers so that each worker performs each task with same asymptotic frequency

*e.g., Monday from 9:10 to 10:30, drive train 9015 from Paris to London with Alice, Bob, or Charlie*
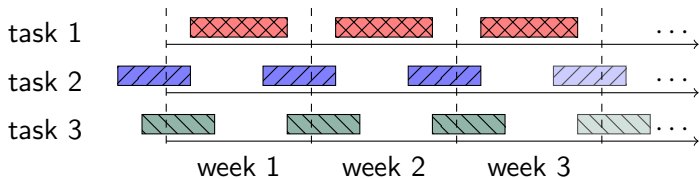
# Feasible assignment



task 1
task 2
task 3

week 1   week 2   week 3

feasible assignment

Alice
Bob
Charlie

week 1   week 2   week 3

# Balanced assignment: an example
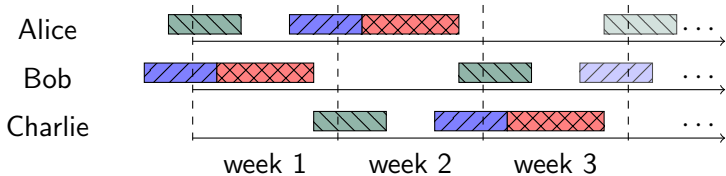
# Balanced assignment: formally

Input: $n$ tasks to be repeated every week, $q$ workers

Assignment: $f \colon [n] \times \mathbb{Z}_{>0} \to [q]$, where $f(i, r) = j$ means worker $j$ performs task $i$ on week $r$

Feasible assignment: $f$ is non overlapping

Balanced assignment:

$$\lim_{t \to +\infty} \frac{1}{t} \big| \{ r \in [t] \colon f(i, r) = j \} \big| = \frac{1}{q} \qquad \forall i, j$$

# Balanced assignments: unachievable case



task 1

task 2

week 1   week 2   week 3

With 2 workers

Alice

Bob

week 1   week 2   week 3

With 3 workers

Alice

Bob

Charlie

week 1   week 2   week 3
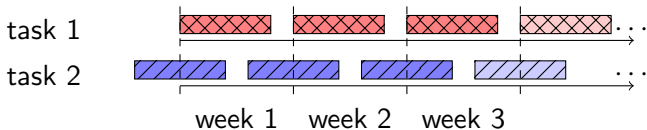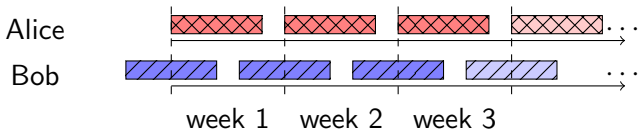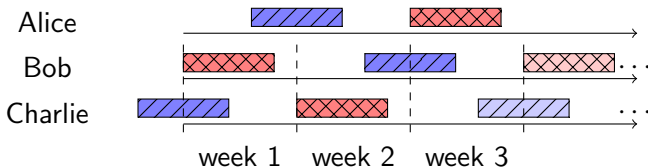
# Main results

**Theorem** G.–Meunier 2024+

There exists a balanced feasible assignment if and only if there exists a feasible assignment with a worker performing each task at least once.

**Theorem** G.–Meunier 2024+

Deciding whether there exists a balanced feasible assignment can be done in polynomial time.

# Moving pebbles

Consider:

- graph $D = (V, A)$ with colors on arcs of vertex disjoint cycles covering $V$
- one pebble on each vertex

Sequence of colors defines sequence of moves of the pebbles

Color: →

## Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.
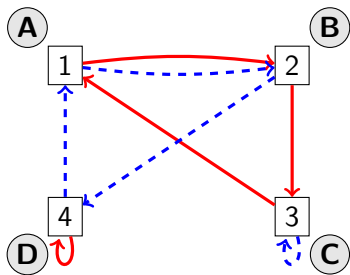


sequence of colors:
$\rightarrow$ -$\rightarrow$ $\rightarrow$ -$\rightarrow$ -$\rightarrow$ $\rightarrow$ -$\rightarrow$ $\rightarrow$

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1-→2 | 2-→4 | 3-→3 | 4-→1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| C | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| D | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.

sequence of colors:

→ -⟩ → -⟩ -⟩ → -⟩ →

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1-⟩2 | 2-⟩4 | 3-⟩3 | 4-⟩1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | **1** | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| B | 0 | **1** | 0 | 0 | 0 | 0 | 0 | 0 |
| C | 0 | 0 | **1** | 0 | 0 | 0 | 0 | 0 |
| D | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 |

# Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.

sequence of colors:
→ -> → -> -> → -> →

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1->2 | 2->4 | 3->3 | 4->1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | 1   | 0   | 0   | 0   | 0    | 1    | 0    | 0    |
| B | 0   | 1   | 0   | 0   | 0    | 0    | 1    | 0    |
| C | 0   | 0   | 1   | 0   | 1    | 0    | 0    | 0    |
| D | 0   | 0   | 0   | 1   | 0    | 0    | 0    | 1    |

# Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.

sequence of colors:
→ -› ➜ -› -› → -› →

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1-›2 | 2-›4 | 3-›3 | 4-›1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | 1   | 0   | 0   | **1** | 0  | 1    | 0    | 0    |
| B | 0   | 1   | **1** | 0 | 0    | 0    | 1    | 0    |
| C | 0   | **1** | 1 | 0   | 1    | 0    | 0    | 0    |
| D | **1** | 0 | 0   | 1   | 0    | 0    | 0    | 1    |

# Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.

sequence of colors:
$\rightarrow$ -$\rightarrow$ $\rightarrow$ -$\rightarrow$ -$\rightarrow$ $\rightarrow$ -$\rightarrow$ $\rightarrow$

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1-›2 | 2-›4 | 3-›3 | 4-›1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | 1   | 0   | 0   | 1   | 0    | 1    | 0    | **1** |
| B | 0   | 1   | 1   | 0   | **1** | 0    | 1    | 0    |
| C | 0   | 1   | 1   | 0   | 1    | 0    | **1** | 0    |
| D | 1   | 0   | 0   | 1   | 0    | **1** | 0    | 1    |

# Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.

sequence of colors:

→ -> → -> -> → -> →

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1->2 | 2->4 | 3->3 | 4->1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | 1   | 0   | 0   | 1   | **1**| 1    | 0    | 1    |
| B | 0   | 1   | 1   | 0   | 1    | **1**| 1    | 0    |
| C | 0   | 1   | 1   | 0   | 1    | 0    | **2**| 0    |
| D | 1   | 0   | 0   | 1   | 0    | 1    | 0    | **2**|

## Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.

sequence of colors:

→ -› → -› -› ➜ -› →

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1-›2 | 2-›4 | 3-›3 | 4-›1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | 1   | **1** | 0   | 1   | 1    | 1    | 0    | 1    |
| B | 0   | 1   | 1   | **1** | 1    | 1    | 1    | 0    |
| C | 0   | 1   | **2** | 0   | 1    | 0    | 2    | 0    |
| D | **2** | 0   | 0   | 1   | 0    | 1    | 0    | 2    |

# Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.

sequence of colors:

→ -> → -> -> → -> →

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1->2 | 2->4 | 3->3 | 4->1 |
|---|-----|-----|-----|-----|------|------|------|------|
| A | 1   | 1   | 0   | 1   | 1    | 1    | **1** | 1    |
| B | 0   | 1   | 1   | 1   | 1    | 1    | 1    | **1** |
| C | 0   | 1   | 2   | 0   | **2** | 0   | 2    | 0    |
| D | 2   | 0   | 0   | 1   | 0    | **2** | 0    | 2    |

# Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.
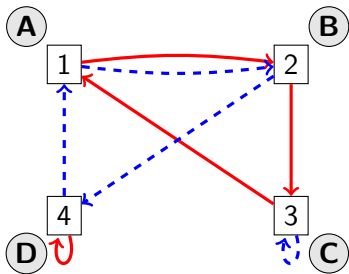
sequence of colors:

$\rightarrow$ $\dashrightarrow$ $\rightarrow$ $\dashrightarrow$ $\dashrightarrow$ $\rightarrow$ $\dashrightarrow$ $\rightarrow$

arcs visiting:

|   | 1→2 | 2→3 | 3→1 | 4→4 | 1⇢2 | 2⇢4 | 3⇢3 | 4⇢1 |
|---|-----|-----|-----|-----|-----|-----|-----|-----|
| A | 1   | 1   | **1** | 1   | 1   | 1   | 1   | 1   |
| B | **1** | 1   | 1   | 1   | 1   | 1   | 1   | 1   |
| C | 0   | **2** | 2   | 0   | 2   | 0   | 2   | 0   |
| D | 2   | 0   | 0   | **2** | 0   | 2   | 0   | 2   |

# Moving pebbles

Aim: Find a sequence of colors making each pebble visit each arc with same asymptotic frequency.
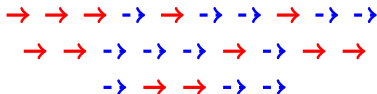


sequence of colors:

→ → → -› → -› -› → -› -›
→ → -› -› -› → -› → →
-› → → -› -›

arcs visiting:

| | 1→2 | 2→3 | 3→1 | 4→4 | 1-›2 | 2-›4 | 3-›3 | 4-›1 |
|---|---|---|---|---|---|---|---|---|
| A | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| B | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| C | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |
| D | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 |

# Back to the results

**Theorem** G.–Meunier 2024+

If the graph is strongly connected, then there exists a sequence of colors making each pebble visit each arc with same asymptotic frequency.

2 proofs:

- Markov chains: non constructive
- Contructive proof with additionnal result:

**Proposition**

If the number of workers $q$ is constant, then a balanced assignment can be computed in polynomial time when it exists.

Upper bound on the period: $q^2q!$

# Thank you